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# BKW-Operators on the Interval and the Sequence Spaces

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Let $\Gamma$ be the closed unit interval or $\Gamma = \{1/n;\, n = 1, 2, ..., \infty\}$. We give a complete characterization of BKW-operators on $C(\Gamma)$ for the test functions $\{1, t, t^2\}$.


## 1. INTRODUCTION

Let $I$ be the closed unit interval $[0, 1]$ and let $C(I)$ be the Banach space of real valued continuous functions on $I$ with the supremum norm. In [2], Korovkin proved the following theorem; if $\{T_n\}_n$ is a sequence of positive operators on $C(I)$ such that $\|T_n t^j - t^j\|_\infty \to 0$ for $j = 0, 1, 2$, then $\{T_n\}_n$ converges strongly to the identity operator; see also [3]. This theorem has been studied from various view points; see the monograph by Altomare and Campiti [1]. Recently as a generalization of the Korovkin theorem, the second author [8–12] studied Bohman–Korovkin–Wulbert-type approximation theorems for more general operators. Let $X$ and $Y$ be normed spaces and let $T \in L(X, Y)$, where $L(X, Y)$ is the space of all bounded linear operators from $X$ into $Y$. Let $S$ be a subset of $X$. Then $T$ is called a BKW-operator for the test functions $S$ if for every net $\{T_\lambda\}_{\lambda \in \Lambda}$ of bounded linear operators in $L(X, Y)$ satisfying

$$\lim_\lambda \|T_\lambda\| = \|T\|, \qquad \lim_\lambda \|T_\lambda f - Tf\|_\infty = 0 \qquad \text{for} \quad f \in S,$$

it follows that $\{T_\lambda\}_\lambda$ converges strongly to $T$ on $X$. We denote by BKW$(X, Y; S)$ the set of BKW-operators for $S$. When $X = Y$, we write



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$L(X)$ and $\mathrm{BKW}(X; S)$. When the set $S$ of functions is given, we are interested in the following problem: Which operators are BKW-operators? In [4], the first author, Takagi, and Watanabe show that when $S$ is a separable subset of $X$, we can replace in the definition of BKW-operators the condition "a net $\{T_\lambda\}_\lambda$" by "a sequence $\{T_n\}_n$." We note that $C(I)$ is a separable Banach space.

It is also interesting to determine all operators in $\mathrm{BKW}\,(C(I); \{1, t, t^2\})$. In [12], the second author gives a characterization of $\mathrm{BKW}(C(I); \{1, t\})$. He also describes operators $T$ in $\mathrm{BKW}(C(I); \{1, t, t^2\})$ which satisfy $T1 = 1$ and $\|T\| = 1$; see also [9]. For such an operator $T$, there exists a continuous function $x(t)$ on $I$ with $0 \leqslant x(t) \leqslant 1$, and there exists an open subset $G$ of $I$ such that $0 < x(t) < 1$ on $G$, $x(t) = 0$ or $1$ for $t \in \partial G$, and

$$(Tf)(t) = \begin{cases} f(x(t)) & \text{if} \quad t \in I \backslash G \\ (1 - x(t))\, f(0) + x(t)\, f(1) & \text{if} \quad t \in G \end{cases}$$

for every $f \in C(I)$, where $\partial G$ denotes the topological boundary of $G$ in $I$. By definition, we have $\mathrm{BKW}(C(I); \{1, t\}) \subset \mathrm{BKW}(C(I); \{1, t, t^2\})$.

The purpose of this paper is to give a complete characterization of operators $T$ in $\mathrm{BKW}(C(I); \{1, t, t^2\})$. In Section 2, we prove that such an operator $T$ has the following form:

$$(Tf)(t) = a(t)\, f(0) + b(t)\, f(1) + c(t)\, f(x(t)).$$

But we cannot expect that $a(t), b(t), c(t)$, and $x(t)$ are continuous. In Section 3, we determine BKW-operators on the sequence $K = \{1/n; n = 1, 2, ..., \infty\}$ for $\{1, t, t^2\}$. To describe these operators, we need one more term in the above form of $T$. The structure of BKW-operators for $\{1, t, t^2\}$ is complicated. Using this characterization, we answer the problems given in [11].

## 2. BKW-OPERATORS ON THE INTERVAL

Let $\Gamma$ be a compact subset of $I$ and let $C(\Gamma)$ be the space of all real continuous functions on $\Gamma$. In this section, we study the case $\Gamma = I$, and in the next section we study the case $\Gamma = K = \{1/n; n = 1, 2, ..., \infty\}$. By the Riesz representation theorem, the dual space of $C(\Gamma)$ can be identified with $M(\Gamma)$ the space of bounded real Borel measures on $\Gamma$. Since $M(\Gamma)$ is the dual space, we can consider the weak*-topology on $M(\Gamma)$. Let $M_1(\Gamma) = \{\mu \in M(\Gamma); \|\mu\| \leqslant 1\}$, where $\|\mu\|$ is the total variation norm of $\mu$. For $\zeta \in \Gamma$, we denote by $\delta_\zeta$ the unit point mass at $\zeta$.

Our theorem is the following.

THEOREM 1. *$T \in \mathrm{BKW}(C(I); \{1, t, t^2\})$ with $\|T\| = 1$ if and only if*

$$(Tf)(t) = a(t)\, f(0) + b(t)\, f(1) + c(t)\, f(x(t)) \tag{\#}$$

*for every $f \in C(I)$, where $a, b, c$, and $x$ are real functions satisfying the following conditions*:

(i) $|a| + |b| + |c| = 1$ *on* $I$.

(ii) $0 \leqslant x \leqslant 1$ *on* $I$, *and if* $x(t_0) = 0$ *or* $1$ *for some* $t_0 \in I$ *then* $c(t_0) = 0$.

(iii) *If* $0 < |c(t_0)| < 1$, *then* $|(a+b+c)(t_0)| < |(a+b)(t_0)| + |c(t_0)| = 1$.

(iv) *If* $0 < |c(t_0)| < 1$ *and* $0 < x(t_0) < 1/2$, *then* $a(t_0) = 0$.

(v) *If* $0 < |c(t_0)| < 1$ *and* $1/2 < x(t_0) < 1$, *then* $b(t_0) = 0$.

(vi) $a(t)\, \delta_0 + b(t)\, \delta_1 + c(t)\, \delta_{x(t)}$, $t \in I$, *moves continuously in* $M_1(I)$ *with the weak*-topology.*

We note that $a, b, c$, and $x$ may not be continuous. The measures $a(t)\, \delta_0 + b(t)\, \delta_1 + c(t)\, \delta_{x(t)}$, $0 \leqslant t \leqslant 1$, are continuous with respect to the weak*-topology. Here we list some of their properties.

(a) If $|c(t_0)| > 0$, then $a, b, c$, and $x$ are continuous on some neighborhood of $t_0$.

(b) $x(t)$ may not be continuous at the point $t_0 \in I$ with $c(t_0) = 0$.

(c) If $t_n \to t_0$ and $x(t_n) \to 0$, then $a(t_n) + c(t_n) \to a(t_0)$.

(d) If $0 < x(t_0) < 1$ and $x$ is continuous at $t_0$, then $a, b, c$ are continuous at $t_0$.

To prove our theorem, we need some lemmas. Let $S_2 = \{1, t, t^2\}$ and let $\tilde{S}_2$ be the closed linear span of $S_2$ in $C(\Gamma)$. We denote by $U_{S_2}(M_1(\Gamma))$ the set of measures $\mu \in M_1(\Gamma)$ such that if $\nu \in M_1(\Gamma)$ and $\int_\Gamma f\, d\mu = \int_\Gamma f\, d\nu$ for every $f \in S_2$, then $\mu = \nu$. $U_{S_2}(M_1(\Gamma))$ is called the uniqueness set for $S_2$. The condition of $\int_\Gamma f\, d\mu = \int_\Gamma f\, d\nu$ for $f \in S_2$ is equivalent to the one of $\int_\Gamma f\, d\mu = \int_\Gamma f\, d\nu$ for $f \in \tilde{S}_2$. By the definition of the uniqueness set, the following lemma is not difficult to prove; see [4, Lemma 4; 12, Lemma 2.1].

LEMMA 1. *If $\mu \in U_{S_2}(M_1(\Gamma))$, then $\|\mu\| = 1$, $-\mu \in U_{S_2}(M_1(\Gamma))$ and $1 = \sup\{|\int_\Gamma f\, d\mu|; f \in \tilde{S}_2, \|f\|_\infty = 1\}$.*

By Hahn–Banach theorem and Riesz representation theorem, we have that $\mu \in U_{S_2}(M_1(\Gamma))$ if and only if the norm of $\mu$ as a linear functional of $\tilde{S}_2$ is 1 and $\mu$ has a unique norm preserving extension to $C(\Gamma)$. The following lemma is proved by Micchelli [5, 6]. Here we give another proof.

LEMMA 2. *Let $\mu$ be a positive measure on $I$ with $\|\mu\| = 1$. Then $\mu \in U_{S_2}(M_1(I))$ if and only if $\mu$ has the form $\mu = \delta_x$ for some $x \in I$ or $\mu = a\delta_0 + (1-a)\,\delta_1$ for some $a$ with $0 < a < 1$.*

*Proof.* Let

$$W = \left\{ \left( \int_I t\, d\sigma, \int_I t^2\, d\sigma \right); \sigma \in M_1(I), \sigma \geqslant 0, \|\sigma\| = 1 \right\}.$$

Then $W$ is a compact convex subset of $I^2$ and

$$W = \{(x, y) \in I^2; x^2 \leqslant y \leqslant x\}.$$

Let $L_1 = \{(x, x^2) \in I^2;\ 0 \leqslant x \leqslant 1\}$ and $L_2 = \{(x, x) \in I^2;\ 0 < x < 1\}$. Then $L_1 \cup L_2 = \partial W$. For $\sigma \in M_1(I)$ with $\sigma \geqslant 0$, $\|\sigma\| = 1$, we have

$$0 \leqslant \left( \int_I t\, d\sigma \right)^2 \leqslant \int_I t^2\, d\sigma \leqslant \int_I t\, d\sigma \leqslant 1.$$

Then $(\int_I t\, d\sigma, \int_I t^2\, d\sigma) \in L_1$ if and only if $\int_I t\, d\sigma = (\int_I t^2\, d\sigma)^{1/2}$. The condition is well known which guarantees the equality in Hölder's inequality; see [7, p. 65]. Using this fact, we get

$$\left( \int_I t\, d\sigma, \int_I t^2\, d\sigma \right) \in L_1 \qquad \text{if and only if } \sigma = \delta_x \text{ for some } x \in I.$$

In the same way, we can prove that

$$\left( \int_I t\, d\sigma, \int_I t^2\, d\sigma \right) \in L_2 \qquad \text{if and only if } \sigma = a\delta_0 + (1-a)\delta_1 \text{ for some } 0 < a < 1.$$

Hence if $\zeta \in L_1 \cup L_2$ then there exists a unique measure $\sigma_\zeta$ in $M_1(I)$ such that

$$\zeta = \left( \int_I t\, d\sigma_\zeta, \int_I t^2\, d\sigma_\zeta \right), \qquad \sigma_\zeta \geqslant 0, \quad \|\sigma_\zeta\| = 1.$$

Therefore,

$$\text{if } \zeta \in L_1 \quad \text{then } \sigma_\zeta = \delta_x \text{ for some } 0 \leqslant x \leqslant 1, \tag{1}$$

$$\text{if } \zeta \in L_2 \quad \text{then } \sigma_\zeta = a\delta_0 + (1-a)\delta_1 \text{ for some } 0 < a < 1. \tag{2}$$

By the properties of $W$, we see that if $\xi$ is an interior point of $W$ then there are infinitely many representation of $\xi$,

$$\xi = a\zeta_1 + (1-a)\,\zeta_2, \qquad \zeta_1 \in L_1, \quad \zeta_2 \in L_2, \quad 0 < a < 1.$$

This means that there are infinitely many measures $\sigma \in M_1(I)$ such that

$$\xi = \left( \int_I t \, d\sigma, \int_I t^2 \, d\sigma \right), \qquad \sigma \geqslant 0, \quad \|\sigma\| = 1.$$

Now let $\mu \in M_1(I)$ with $\mu \geqslant 0$. Put $\zeta = (\int_I t \, d\mu, \int_I t^2 \, d\mu)$. Then by the above observation and the definition of $U_{S_2}(M_1(I))$, $\mu \in U_{S_2}(M_1(I))$ if and only if $\zeta \in L_1 \cup L_2$. Hence by (1) and (2), we have our assertion.

Generally we have the following.

Lemma 3. *$\mu \in U_{S_2}(M_1(I))$ if and only if $\mu$ has one of the following forms*:

(i) $\mu = \pm \delta_x$, $0 \leqslant x \leqslant 1$,

(ii) $\mu = a\delta_0 + b\delta_1$, $|a| + |b| = 1$,

(iii) $\mu = a\delta_x + b\delta_1$, $|a+b| < |a| + |b| = 1$ *and* $0 < x < 1/2$,

(iv) $\mu = a\delta_0 + b\delta_x$, $|a+b| < |a| + |b| = 1$ *and* $1/2 < x < 1$,

(v) $\mu = a\delta_0 + b\delta_1 + c\delta_{1/2}$, $\quad |a+b+c| < |a+b| + |c| = |a| + |b| + |c| = 1$.

*Proof.* First, suppose that $\mu \in U_{S_2}(M_1(I))$. By Lemma 1, $\|\mu\| = 1$ and

$$1 = \sup \left\{ \left| \int_0^1 f \, d\mu \right| ; f \in \widetilde{S}_2, \|f\|_\infty = 1 \right\}.$$

Since the unit sphere of $\widetilde{S}_2$ is compact, there exists $f_0$ in $\widetilde{S}_2$ such that

$$\left| \int_0^1 f_0 \, d\mu \right| = 1 \qquad \text{and} \qquad \|f_0\|_\infty = 1. \tag{3}$$

Write $f_0(t) = a_0 + a_1 t + a_2 t^2$. Then one of the following cases happens:

$$\{t \in I; |f_0(t)| = 1\} = I, \tag{4}$$

$$\{t \in I; |f_0(t)| = 1\} = \{x\}, \qquad 0 \leqslant x \leqslant 1, \tag{5}$$

$$\{t \in I; |f_0(t)| = 1\} = \{0, 1\}, \tag{6}$$

$$\{t \in I; |f_0(t)| = 1\} = \{x, 1\}, \qquad 0 < x < 1/2, \tag{7}$$

$$\{t \in I; |f_0(t)| = 1\} = \{0, x\}, \qquad 1/2 < x < 1, \tag{8}$$

$$\{t \in I; |f_0(t)| = 1\} = \{0, 1/2, 1\}. \tag{9}$$

Suppose that (4) happens. Then $f_0$ is a constant function and $f_0 = 1$ or $-1$. Hence by (3), $\mu \geqslant 0$ or $\mu \leqslant 0$. By Lemma 1, we may assume that $\mu \geqslant 0$. By Lemma 2, $\mu$ has one of the forms in (i) or (ii).

It is easy to see that the case (5) yields (i) and the case (6) yields (ii).

Suppose that (7) happens. In this case, we have $f_0(x)\, f_0(1) = -1$. If $\mu(\{x\}) = 0$ or $\mu(\{1\}) = 0$ then $\mu$ has the form (i). If $0 < |\mu(\{x\})| < 1$, then $\mu$ has the form (iii). In the same way, the case (8) yields (i) or (iv).

Suppose that (9) happens. Then $f_0(0) = f_0(1)$ and $f_0(0)\, f_0(1/2) = -1$. Hence $\mu$ has one of the forms (i), (ii) or (v).

Next, we prove that if a measure $\mu$ satisfies one of the conditions (i)–(v), then $\mu \in U_{S_2}(M_1(I))$. The proof is almost the same for every case, so that we only prove the following two cases:

(iii′) $\mu = a\delta_x - b\delta_1$, $a, b > 0$, $a + b = 1$, and $0 < x < 1/2$,

(v′) $\mu = a\delta_0 + b\delta_1 - c\delta_{1/2}$, $a, b, c > 0$, and $a + b + c = 1$.

Suppose that $\mu$ has the form (iii′). Let $v \in M_1(I)$ such that $\int_0^1 f\, d\mu = \int_0^1 f\, dv$ for every $f \in \tilde{S}_2$. Put

$$g(t) = -2(t-x)^2/(1-x)^2 + 1, \qquad t \in I.$$

Then $g \in \tilde{S}_2$, $\|g\|_\infty = 1$, $g(x) = 1$, $g(1) = -1$, and

$$\int_0^1 g\, dv = \int_0^1 g\, d\mu = a + b = 1.$$

Hence $v$ has the form $v = a'\delta_x - b'\delta_1$. Since $\int_0^1 t^j\, d\mu = \int_0^1 t^j\, dv$ for $j = 0$ and 1, we have $a - b = a' - b'$ and $ax - b = a'x - b'$. Therefore $a = a'$ and $b = b'$, so that $\mu = v$. This means that $\mu \in U_{S_2}(M_1(I))$.

Next, suppose that $\mu$ has the form (v′). Let $v \in M_1(I)$ such that $\int_0^1 f\, d\mu = \int_0^1 f\, dv$ for every $f \in \tilde{S}_2$. Put $h(t) = 8t^2 - 8t + 1$. Then $h \in \tilde{S}_2$, $\|h\| = 1$, $h(0) = h(1) = 1$, $h(1/2) = -1$, and

$$\int_0^1 h\, dv = \int_0^1 h\, d\mu = a + b + c = 1.$$

Hence $v$ has the form $v = a'\delta_0 + b'\delta_1 - c'\delta_{1/2}$. Since $\int_0^1 t^j\, d\mu = \int_0^1 t^j\, dv$ for $j = 0, 1, 2$, we have

$$a + b - c = a' + b' - c', \qquad b - c/2 = b' - c'/2, \qquad b - c/4 = b' - c'/4.$$

Therefore $a = a'$, $b = b'$, and $c = c'$, so that we obtain $\mu \in U_{S_2}(M_1(I))$.

Let $T \in L(C(\Gamma))$ with $\|T\| = 1$. We denote by $T^*$ the dual operator of $T$. Then for $t \in \Gamma$ we have $\int_\Gamma f\, dT^*\delta_t = (Tf)(t)$ for $f \in C(\Gamma)$. Hence $T^*\delta_t$, $t \in \Gamma$, are continuous with respect to the weak*-topology. The following lemma is proved in [12].

LEMMA 4. *Let* $T \in L(C(T))$. *Then* $T \in \mathrm{BKW}(C(\Gamma); S_2)$ *with* $\|T\| = 1$ *if and only if* $T^*\delta_t \in U_{S_2}(M_1(\Gamma))$ *for every* $t \in \Gamma$.

*Proof of Theorem* 1. Suppose that $T$ is of the form $(\#)$, and $a$, $b$, $c$, and $x$ satisfy conditions (i)–(vi). By (vi), we have $Tf \in C(I)$, so that $T \in L(C(I))$ and $\|T\| = 1$. To show $T \in \mathrm{BKW}(C(I); S_2)$, we use Lemma 4. Let $t \in I$. Then by $(\#)$, we have

$$T^*\delta_t = a(t)\,\delta_0 + b(t)\,\delta_1 + c(t)\,\delta_{x(t)}.$$

By conditions (i)–(v), we know that the measure $T^*\delta_t$ has one of the forms in Lemma 3. Hence $T^*\delta_t \in U_{S_2}(M_1(I))$. Therefore by Lemma 4, we have $T \in \mathrm{BKW}(C(I); S_2)$.

To prove the converse, let $T \in \mathrm{BKW}(C(I); S_2)$. For each $t \in I$, by Lemma 4 we have $T^*\delta_t \in U_{S_2}(M_1(I))$, and by Lemma 3, $T$ has the form $(\#)$ and all conditions (i)–(vi) are satisfied.

## 3. BKW-OPERATORS ON THE SEQUENCE SPACE

Let

$$K = \{1/n;\, n = 1, 2, ..., \infty\},$$

where we use the convention $1/\infty = 0$. Then $K$ is a compact subset of $I$ and $C(K)$ is isomorphic to the space of real convergent sequences. In this section, we determine all operators in $\mathrm{BKW}(C(K); \{1, t, t^2\})$. According to Theorem 1, the reader may suspect that such an operator $T$ has the form

$$(Tf)(1/n) = a(1/n)\, f(0) + b(1/n)\, f(1) + c(1/n)\, f(x(1/n)), \qquad x(1/n) \in K.$$

But there are some other possibilities.

THEOREM 2. $T \in \mathrm{BKW}(C(K); \{1, t, t^2\})$ *with* $\|T\| = 1$ *if and only if*

$$(Tf) = a(t)\, f(0) + b(t)\, f(1) + c(t)\, f(x(t)) + d(t)\, f(y(t)), \qquad t \in K, \qquad (\#\#)$$

*for every* $f \in C(K)$, *where* $a$, $b$, $c$, $d$, $x$ *and* $y$ *are real functions on* $K$ *satisfying the following conditions*:

(i) $|a| + |b| + |c| + |d| = 1$ *on* $K$.

(ii) $x(K) \subset K$, $y(K) \subset K$, $x \leqslant y$ *on* $K$, *and if* $x(t_0) = 0$ *or* $1$, $t_0 \in K$, *then* $c(t_0) = 0$.

(iii) *There exist subsets* $K_1$ *and* $K_2$ *of* $K$ *with* $K_1 \cup K_2 = K$ *and* $K_1 \cap K_2 = \varnothing$ *such that* $d = 0$ *on* $K_1$ *and* $0 < x < 1$ *on* $K_2$.

(iv) *If* $0<|c(t_0)|<1$ *and* $t_0\in K_1$, *then* $|(a+b+c)(t_0)|<|(a+b)(t_0)|+|c(t_0)|=1$.

(v) *If* $0<|c(t_0)|<1$, $t_0\in K_1$, *and* $0<x(t_0)<1/2$, *then* $a(t_0)=0$.

(vi) $x(t)^{-1}-y(t)^{-1}=1$ *for every* $t\in K_2$.

(vii) $a=b=0$ *and* $|c+d|=|c|+|d|=1$ *on* $K_2$.

(viii) $a(t)\,\delta_0+b(t)\,\delta_1+c(t)\,\delta_{x(t)}+d(t)\,\delta_{y(t)}$, $t\in K$, *are continuous with respect to the weak*-topology.*

The idea of the proof is the same as the one in Section 2. Theorem 2 follows easily from the next two lemmas. Therefore we give only their proofs and leave the proof of Theorem 2 to the reader.

LEMMA 5. *Let* $\mu$ *be a positive measure on* $K$ *with* $\|\mu\|=1$. *Then* $\mu\in U_{S_2}(M_1(K))$ *if and only if* $\mu=\delta_x$, $x\in K$, $\mu=a\delta_0+(1-a)\delta_1$ *for* $0<a<1$, *or* $\mu=a\delta_{1/n}+(1-a)\,\delta_{1/(n+1)}$ *for* $0\leqslant a\leqslant 1$ *and* $1\leqslant n<\infty$.

*Proof.* The idea of the proof is the same as the one of Lemma 2. Let

$$V=\left\{\left(\int_K t\,d\sigma,\int_K t^2\,d\sigma\right);\sigma\in M_1(K),\sigma\geqslant 0,\ \|\sigma\|=1\right\}.$$

Then $V$ is a compact subset of $I^2$ and coincides with the convex hull of the set

$$\{(1/n,1/n^2);n=1,2,...,\infty\}.$$

Let $L_{i,j}$ be the line segment connecting points $(1/i,1/i^2)$ and $(1/j,1/j^2)$ for $1\leqslant i<j\leqslant\infty$. Then the boundary $\partial V$ of $V$ coincides with $L_{1,\infty}\cup(\bigcup_{i=1}^{\infty}L_{i,i+1})$. For each point $\zeta\in\partial V$ there exists a unique measure $\sigma_\zeta$ in $M_1(K)$ such that

$$\zeta=\left(\int_K t\,d\sigma_\zeta,\int_K t^2\,d\sigma_\zeta\right),\qquad \sigma_\zeta\geqslant 0,\quad \|\sigma_\zeta\|=1.$$

Moreover, if $\zeta\in L_{i,i+1}$ then $\sigma_\zeta=b\delta_{1/i}+(1-b)\,\delta_{1/(i+1)}$, $0\leqslant b\leqslant 1$, and if $\zeta\in L_{1,\infty}$ then $\sigma_\zeta=b\delta_0+(1-b)\,\delta_1$, $0\leqslant b\leqslant 1$. Also, if $\zeta$ is an interior point of $V$, then there exist many measures $\nu\in M_1(K)$ such that

$$\zeta=\left(\int_K t\,d\nu,\int_K t^2\,d\nu\right),\qquad \nu\geqslant 0,\quad \|\nu\|=1.$$

Let $\mu\in M_1(K)$ with $\mu\geqslant 0$ and $\|\mu\|=1$. Then we have that $\mu\in U_{S_2}(M_1(K))$ if and only if $(\int_K t\,d\mu,\int_K t^2\,d\mu)\in\partial V$ by the above observation and the definition of the uniqueness set. Therefore we get the desired assertion.

LEMMA 6. *Let $\mu$ be a measure on K with $\|\mu\|=1$. Then $\mu \in U_{S_2}(M_1(K))$ if and only if $\mu$ has one of the following forms*:

(i) $\mu = \pm\delta_{1/n}$, $n=1, 2, ..., \infty$,

(ii) $\mu = a\delta_0 + b\delta_1$, $|a|+|b|=1$,

(iii) $\mu = a\delta_{1/n} + b\delta_1$, $|a+b|<|a|+|b|=1$ *and* $1<n<\infty$,

(iv) $\mu = a\delta_0 + b\delta_1 + c\delta_{1/2}$, $|a+b+c|<|a+b|+|c|=|a|+|b|+|c|=1$,

(v) $\mu = a\delta_{1/n} + b\delta_{1/(n+1)}$, $|a+b|=|a|+|b|=1$ *and* $1\leqslant n<\infty$.

*Proof.* Since $K\subset I$, we have $M_1(K)\subset M_1(I)$. By the definition of the uniqueness set for $S_2$, we have

$$M_1(K)\cap U_{S_2}(M_1(I))\subset U_{S_2}(M_1(K)). \tag{1}$$

By Lemma 3, if $\mu\in M_1(K)\cap U_{S_2}(M_1(I))$ then $\mu$ has one of the forms (i)–(iv) in Lemma 6. Since $\{1/2<x<1\}\cap K=\emptyset$, case (iv) in Lemma 3 does not happen. The measure in (v) has the form as $a\delta_{1/n}+b\delta_{1/(n+1)}$, and, moreover, if $a=0$ or $b=0$ then this measure has the form in (i). Hence by (1) we need to prove that

$$\mu\in U_{S_2}(M_1(K))\backslash(M_1(K)\cap U_{S_2}(M_1(I))) \tag{2}$$

if and only if there exists $n$ with $1\leqslant n<\infty$ such that

$$\mu = a\delta_{1/n}+b\delta_{1/(n+1)}, \qquad |a+b|=|a|+|b|=1, \qquad a\neq 0, \text{ and } b\neq 0. \tag{3}$$

First, suppose that $\mu$ satisfies (2). Since $\mu\in U_{S_2}(M_1(K))$, by the same way as the first paragraph of the proof of Lemma 3, there exists $f_0$ in $\tilde{S}_2$ such that

$$\left|\int_K f_0\, d\mu\right|=1, \qquad \|f_0\|_\infty=1.$$

Here it can happen that $f_0$ is constant or $f_0$ is nonconstant on $K$. When $f_0$ is nonconstant, by the same argument as in the proof of Lemma 3, $\mu$ has one of the form in (i)–(iv). Hence $\mu\in M_1(K)\cap U_{S_2}(M_1(I))$, so that $f_0$ must be constant. Then by Lemma 5, $\mu$ has the forms

$$\mu=\pm\delta_{1/n}, \qquad \mu=\pm(a\delta_0+(1-a)\delta_1),$$
$$\text{or} \qquad \mu=\pm(a\delta_{1/n}+(1-a)\,\delta_{1/(n+1)}).$$

Since $\mu\notin M_1(K)\cap U_{S_2}(M_1(I))$, $\mu$ has the form in (3) for some $1\leqslant n<\infty$.

The implication (3) $\Rightarrow$ (2) follows from Lemmas 1, 3, and 5.

In [11], the second author studied the following operators on $C(K)$. For each fixed positive integer $m$, let

$$(T_m f)(1/n) = \begin{cases} -(f(1)+f(1/m))/2 & \text{if} \quad n=1 \\ f(1/(n-1)) & \text{if} \quad 2 \leqslant n \leqslant \infty, \end{cases}$$

$$(U_m f)(1/n) = \begin{cases} (f(1)-f(1/m))/2 & \text{if} \quad n=1 \\ f(1/(n-1)) & \text{if} \quad 2 \leqslant n \leqslant \infty. \end{cases}$$

He proved that $T_1$, $T_2$, $U_m$, $m \geqslant 2$, are contained in BKW($C(K)$; $\{1, t, t^2\}$). He asked whether $T_m$, $m \geqslant 3$, and $U_1$ are BKW-operators or not. We note that $U_1$ is the unilateral shift operator on $C(K)$. As application of Theorem 2, we have that $T_m$, $m \geqslant 3$, and $U_1$ are not BKW-operators for $S_2$. Also the backward shift operator defined by

$$(Bf)(1/n) = f(1/(n+1)) \qquad \text{for} \quad f \in C(K)$$

and the operator defined by

$$(Tf)(1/n) = (f(1/n) + f(1/(n+1)))/2$$

are BKW-operators for $S_2$.

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